Randomized Constructions of Expanders

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1 Introduction

Expanders are graphs in which small sets of vertices have large neighborhoods.

Definition 1.1 ([12])**.** We say a graph G is a *c-expander* if, for all $S \subseteq V(G)$ where $|S| < |V(G)|/2$, we have $|\partial S| \geq c|S|$.

In practice, we do not care too much about the constant c. As a result, we often work with a more convenient reformulation:

Definition 1.2 ([12])**.** We say a graph G is a *spectral λ-expander* if the second-largest (in magnitude) eigenvalue of its normalized adjacency matrix, denoted λ_2, is at most λ.

Cheeger's Inequality [7] implies that these definitions are equivalent up to a change in the constant. We will mostly use the latter definition in this paper.

Expanders have numerous properties with computer science applications. Expanders are highly connected, making them ideal for designing efficient digital networks [20] and neural net architectures [18]. Moreover, random walks on expanders are known to converge rapidly [8], so they can be used to construct near-perfect hash functions [6]. In addition, the sets of neighbors of vertices in an expander can induce good error-correcting codes [19] due to their uniformity.

Pinsker [17] gave the earliest construction of a family of expander graphs using only the probabilistic method. Bollobas [5] and Friedman [10] later used similar elementary techniques to show that random regular graphs are expanders with high probability. Other constructions have utilized more advanced mathematical machinery. Margulis [15] and Alon and Roichman [1] show that certain Cayley Graphs, i.e. graphs based on group theoretic relations, are likely expanders. Lubotzky et al. [14] construct *Ramanjuan graphs*, which achieve the best possible spectral expansion, using deep number theory; moreover, their construction can be derandomized.

We aim to provide an overview of randomized constructions of expanders. We will discuss examples of results covering three broad techniques – algebraic, combinatorial, and geometric. We include our own progress towards understanding a special case in Section 3. We additionally include our empirical observations about the models studied in Section 5.

2 Algebraic Construction

In this section, we focus on an algebraic construction of Expander provided by Alon and Roichman [1].

Definition 2.1. For a group G and a set of elements $S \subset G$, The Cayley graph $X(G, S)$ is an undirected graph whose set of vertices is elements of G, and there exists an edge between g and gs for any $g \in G, s \in S$.

In [4], it is proven that the diameter is almost surely logarithmic for a random Cayley graph with $|S| = c \log n$ elements. Since δ-expanders also have logarithmic diameters, we are curious if random Cayley graphs are expanders.

Theorem 2.2. *For any $0 < \delta < 1$, there exists a $c(\delta) > 0$ such that the following holds. Let G be a group of order n, and let set S be a random set of $c(\delta) \log n$ elements. Then*

$$\mathbb{E}[\lambda_2(X(G, S))] < 1 - \delta.$$

Corollary 2.3. *For any $0 < \delta < 1$, there exists a $c(\delta) > 0$ such that the following holds. Let G be a group of order n, and let set S be a random set of $c(\delta) \log n$ elements, the Cayley Graph $X(G, S)$ is an δ-expander with probability $1 - o(1)$.*

Lemma 2.4. *Let P_{2m} be the probability of a random walk of length $2m$ to be closed in the Cayley graph $X(G,S)$. Let $|S| = c\log n$, for any integer l, we have*

$$\mathbb{E}[P_{2m}] \leq 2^{2m}(2/d)^l + 2^{2m-2l}(m/d)^{m-l} + 1/n + O(m/n^2).$$

Proof. We can consider a random walk of length $2m$ to be generated by the following process:

- We choose in the free monoid $M_{2|S|}$, generated by $|S|$ distinct letters and their inverses, a random word of length $2m$.
- We reduce it in the free group $F_{|S|}$.
- We map the letters to distinct elements of G at random.

Then

A The reduced word has length less than $2m - 2l$ in the free group.

B The reduced word has length no less than $2m - 2l$, and no letters appear exactly once in the word.

C None of the above, and the word gets reduced to the identity in group G.

The probability that the random walk is closed is no more than $\Pr[\mathbf{A}] + \Pr[\mathbf{B}] + \Pr[\mathbf{C}]$. We will show that this probability is small. □

Claim 2.5. *Let U be a word of length $2t$ in a free monoid M_{2d}. The probability that U is the identity is no more than $(2/d)^t$.*

Proof. Let $a_1, a_2, \ldots, a_d, a_1^{-1}, a_2^{-1}, \ldots, a_d^{-1}$ be the generators of the free monoid. We will prove this claim by induction, $t = 1$ case is trivial as the probability is $1/2d$. For $t+1$ case, the last cancelling pair of elements can be $(1, 2(t+1))$,$(1,2)$ or $(2t+1, 2t+2)$. This gives $3/2d < 2/d$ probability. Thus, the probability is no more than $(2/d)^t$. □

Consider the subset of letters in the word that was reduced to the identity. The number of possible subsets is at most 2^{2m}, and the length is at least $2l$, we get

$$\Pr[\mathbf{a}] \leq 2^{2m}(2/d)^l.$$

For the second part, let $L > 2(m-l)$ be the length of the reduced word. The number of distinct letters is at most $L/2$. Let S' be the set of first occurrences of each letter. There are at most 2^L possible S'. For the other letters, the probability that they have appeared previously is at most $L/2d$. Thus,

$$\Pr[\mathbf{b}] \leq 2^L(L/2d)^{L/2} \leq 2^{2(m-l)}(m/d)^{m-l}.$$

For the third part, we can only look at the unique letter in the reduced word. This is equivalent to finding the probability of

$$gxh = 1$$

conditioned on x not equal to any other letters in the sequence. Thus, the probability is at most $\frac{1}{n-2m}$, and

$$\Pr[\mathbf{c}] \leq \frac{1}{n-2m} = \frac{1}{n} + O(m/n^2).$$

Now, we can prove the original theorem using the lemma.

Proof of Theorem 2.2. Suppose A is a real matrix with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then for $m \in \mathbb{Z}$:

$$\mathrm{Tr}(A^{2m}) = \sum_{i=1}^{n} \lambda_i^{2m} \geq \lambda_1^{2m} + \lambda_2^{2m}.$$

This implies that

$$\lambda_2 \leq \left(\mathrm{Tr}(A^{2m}) - \lambda_1^{2m}\right)^{1/2m}.$$

By Jensen's inequality, we have

$$\mathbb{E}[\lambda_2] \leq \left(\mathbb{E}[\mathrm{Tr}(A^{2m})] - 1\right)^{1/2m}.$$

Let P_{2m} be the probability of a random walk of length $2m$ to be closed. Now consider A as the normalized adjacency matrix of $X(G,S)$, then $\mathbb{E}[\mathrm{Tr}(A^{2m})] = n\mathbb{E}[P_{2m}]$. Take $2m = b\log n$, by Lemma 2.4, we can take appropriate constant b and c so that the quantity is no more than $1-\delta$. We omit the details of the calculation process, since it is hand-wavy and not the main part of our proof. □

To prove the corollary, we will prove that the second largest eigenvalue (in absolute value) of the adjacency matrix of Cayley graph is concentrated around its mean.

Lemma 2.6. *Let G be an abelian group and S be a set of d elements. Let $\lambda_2 = \lambda_2(X(G,S))$, then*

$$\Pr\left[|\lambda_2 - \mathbb{E}[\lambda_2]| \geq 2c\sqrt{d}\right] \leq 4\exp(-c^2/2).$$

Proof. We can label the elements in S as $x_1, x_2, \ldots, x_d$. Let F_i be the expected value of λ_2 after revealing $x_1, x_2, \ldots, x_i$ in S, it is clear that the sequence $F_1, F_2, \ldots, F_n$ forms a martingale. Each time we reveal some x_i, the degree of each element $g \in G$ increases by 2. Let B be the adjacency matrix of these new edges, we have $\|B\| = 2$. Thus,

$$|F_i - F_{i+1}| \leq 2.$$

Therefore, we can apply the Azuma-Hoeffding Inequality and get

$$\Pr\left[|\lambda_2 - \mathbb{E}[\lambda_2]| \geq 2c\sqrt{d}\right] = \Pr\left[|F_n - F_0| \geq 2c\sqrt{d}\right] \leq 2\exp(-c^2/2).$$

□

By the concentration bound, we know that the random Cayley Graph has the second largest eigenvalue less than $1-\delta$ almost surely. By Cheeger's Inequality, a spectral gap implies that the Graph is an expander. Thus, the corollary holds.

Next, we move on to the Cayley Graph of Abelian groups to show that the $\log n$ factor in Theorem 2.2 is necessary for every Abelian group.

Theorem 2.7. *For every fixed $\delta > 0$ and a finite abelian group G. If the Cayley graph $X(G,S)$ is a δ-expander, then $|S| \geq c\log n$.*

Proof. Suppose $|S| = d$. Let D be the diameter of the Cayley graph. Then every element of the group G can be expressed as follows:

$$s_1^{a_1} s_2^{a_2} \cdots s_d^{a_d},$$

where $S = \{s_1, s_2, \ldots, s_n\}$ and

$$\sum_{i=1}^{d} |a_i| \leq D$$

. There are at most 2^d ways to select the signs of each a_i, and $\binom{D+d}{d}$ ways to choose the absolute values $|a_i|$. This gives

$$2^d \binom{D+d}{d} \geq n.$$

Taking logarithm on both sides, we get

$$\implies d + (D+d)\log(D+d) - D\log D - d\log d \geq \log n.$$

If $X(G,S)$ is a δ expander, the diameter must be $O(\log n)$. This implies that $d = \Omega(\log n)$. □

In conclusion, this paper provides an algebraic way of constructing expanders using random Cayley graphs of a group. This construction is especially tight for abelian groups.

3 Lifting Construction

Definition 3.1. Let G and $\tilde{G}$ be graphs, and let $\pi\colon \tilde{G} \to G$ be a graph homomorphism. We say π is a *covering* if for each $\tilde{v} \in G$, the set of edges with endpoint $\tilde{v}$ is mapped bijectively to the set of edges with endpoint $\pi(\tilde{v})$.

Given a connected graph G and natural number n, a random finite covering of G with fibre size n is given as follows:

- Orient the edges of G arbitrarily.
- Assign a uniformly random permutation $\sigma_e \in \mathbb{S}_n$ to each edge $e \in E(G)$.
- Let $\tilde{G}$ be the graph with vertex set $V(G)\times[n]$ and with an edge connecting (u,i) to $(v,\sigma_e(i))$ for $e = uv \in E(G)$ (note that orientation matters here; however, for the distribution of coverings, it does not, since reversing an edge corresponds to taking the inverse permutation).

Coverings obtained in this way are called *random n-lifts*. In [2], Amit and Linial analyze the expansion properties of random lifts of arbitrary graphs. Asymptotic behavior of the entropy function, $H(x) = -x\log x-(1-x)\log(1-x)$, is crucial for this analysis. The following lemma gives a few properties which will be used.

Lemma 3.2. *The following statements hold for the entropy function:*

(a) *For integers n, m we have $\binom{n}{m} \le \exp(nH(\frac{m}{n})) \le (\frac{ne}{m})^m$.*

(b) *For integers n, m we have $\log\binom{n}{m} = nH(\frac{m}{n})(1-o_n(1))$.*

(c) *For $0 \le x \le \frac{1}{4}$, we have $H(\frac{1}{1+x}) \le H(x) \le H(2x)$.*

Proof. Statements (a) and (b) are well-known and follow, for example, from Stirling's approximation. Statement (c) follows from the fact that $H(x)$ is increasing for $x \le \frac{1}{2}$. For $x \le \frac{1}{4}$, we have $\frac{1}{1+x} \le x \le 2x$, giving the needed inequality. $\square$

Theorem 3.3. *Let G be a connected graph with $|E| > |V|$. Then there is a positive constant ξ (depending on G) such that, as $n \to \infty$, almost every random n-lift of G has edge expansion at least ξ.*

First, we need a lemma. Given permutations σ_1, $\sigma_2 \in \mathbb{S}_n$ and an $\eta > 0$, a set $A \subseteq [n]$ is called *η-bad* if $0 < |A| \le \frac{2n}{3}$ and $|A \cup \sigma_1(A) \cup \sigma_2(A)| \le (1+\eta)|A|$.

Lemma 3.4. *There exists $\varepsilon > 0$ such that, for two uniformly chosen σ_1, $\sigma_2 \in \mathbb{S}_n$, as $n \to \infty$ there are almost no ε-bad sets.*

Proof. For fixed A of size m, the probability that A is ε-bad is at most

$$\binom{n-m}{\lfloor \varepsilon m\rfloor}\left(\frac{\binom{\lfloor(1+\varepsilon)m\rfloor}{m}}{\binom{n}{m}}\right)^2,$$

so we need to bound

$$\sum_{m=1}^{2n/3} B(m)$$

where

$$B(m) = \binom{n}{m}\binom{n-m}{\lfloor \varepsilon m\rfloor}\left(\frac{\binom{\lfloor(1+\varepsilon)m\rfloor}{m}}{\binom{n}{m}}\right)^2.$$

We bound this sum for arbitrary $\varepsilon < \frac{1}{4}$, and will find that for sufficiently small ε, the desired inequality holds. We split into two regimes: $m > \frac{n}{10}$ and $m \le \frac{n}{10}$.

For large m, we may bound $B(m)$ using Lemma 3.2. Setting $\mu = \frac{m}{n}$, we obtain

$$\begin{aligned}
\log B(m) &= \log\left(\binom{n-m}{\lfloor \varepsilon m \rfloor}\binom{\lfloor (1+\varepsilon)m \rfloor}{m}^2 \Big/ \binom{n}{m}\right) \\
&\leq \log\left(\binom{n-m}{\varepsilon m}\binom{(1+\varepsilon)m}{m}^2 \Big/ \binom{n}{m}\right) \\
&\leq \left((n-m)H\left(\frac{\varepsilon m}{n-m}\right) + 2(1+\varepsilon)mH\left(\frac{1}{1+\varepsilon}\right) - nH\left(\frac{m}{n}\right)\right)(1+o_n(1)) \\
&= n\left((1-\mu)H\left(\frac{\varepsilon\mu}{1-\mu}\right) + 2(1+\varepsilon)\mu H\left(\frac{1}{1+\varepsilon}\right) - H(\mu)\right)(1+o_n(1)),
\end{aligned}$$

where we can drop the floor signs because m is sufficiently large. Since $\mu \leq \frac{2}{3}$ we have $\frac{\mu}{1-\mu} \leq 2$. Also, $2(1+\varepsilon) \leq \frac{5}{2}$. Continuing to bound using Lemma 3.2 and the fact that $H(x)$ is increasing for small x we obtain

$$\begin{aligned}
\frac{1}{n}\log B(m) &= \left((1-\mu)H\left(\frac{\varepsilon\mu}{1-\mu}\right) + 2(1+\varepsilon)\mu H\left(\frac{1}{1+\varepsilon}\right) - H(\mu)\right)(1+o_n(1)) \\
&\leq \left((1-\mu)H(2\varepsilon) + \frac{5}{2}\mu H(2\varepsilon) - H(\mu)\right)(1+o(1)) \\
&= \left(\left(1+\frac{3}{2}\mu\right)H(2\varepsilon) - H(\mu)\right)(1+o(1)).
\end{aligned}$$

For small enough ε we can guarantee $(1+\frac{3}{2}\mu)(H(2\varepsilon)) < \min\{H(\frac{1}{10}, \frac{2}{3})\}$ because $H(x)$ approaches 0 as x approaches 0. Thus this part of the sum is negligible.

For $m \leq \frac{1}{10}$, we have

$$\begin{aligned}
\binom{5m/4}{m} &= \binom{5m/4}{m/4} \\
&\leq (5e)^{m/4} \\
&\leq 5^{m/2}.
\end{aligned}$$

using Lemma 3.2. Thus

$$\begin{aligned}
B(m) &= \binom{n-m}{\lfloor \varepsilon m \rfloor}\binom{\lfloor (1+\varepsilon)m \rfloor}{m}^2 \Big/ \binom{n}{m} \\
&\leq \binom{n-m}{\lfloor \varepsilon m \rfloor}\binom{(1+\varepsilon)m}{m}^2 \Big/ \binom{n}{m} \\
&\leq \binom{n-m}{\lfloor \varepsilon m \rfloor} 5^m \Big/ \binom{n}{m}.
\end{aligned}$$

Now for $m < \frac{1}{\varepsilon}$, we have $\lfloor \varepsilon m \rfloor = 0$, in which case $B(m) = O(\frac{1}{n})$. The number of such m does not depend on n, so the contribution from these terms is negligible. For larger m, we have

$$\binom{n}{\lfloor \varepsilon m \rfloor} \leq \left(\frac{ne}{\lfloor \varepsilon m \rfloor}\right)^{\lfloor \varepsilon m \rfloor} \leq \left(\frac{2ne}{\varepsilon m}\right)^{\varepsilon m}$$

using Lemma 3.2 and the fact that $\lfloor \varepsilon m \rfloor \leq \varepsilon m \leq 2\lfloor \varepsilon m \rfloor$. Then using the trivial bound $\binom{n}{m} \geq (\frac{n}{m})^m$, we have

$$\begin{aligned}
B(m) &\leq \left(\frac{5\left(\frac{2ne}{\varepsilon m}\right)^{\varepsilon}}{\frac{n}{m}}\right)^m \\
&= \left(5\left(\frac{2e}{\varepsilon}\right)^{\varepsilon}\left(\frac{n}{m}\right)^{\varepsilon-1}\right)^m \\
&\leq \left(10\left(\frac{n}{m}\right)^{\varepsilon-1}\right)^m
\end{aligned}$$

Because the function $x \mapsto (\frac{n}{x})^x$ is increasing in x for $0 < x \le \frac{1}{e}$ (this can be seen by taking derivatives or using the W-function), we have

$$B(m) \le \left(10 \cdot 10^{\varepsilon - 1}\right)^{n/10}.$$

For small enough ε we have $10^{\varepsilon} < 1$, so that $B(m) = o(1/n)$. Thus the contribution of the terms with $1 \le m \le \frac{n}{10}$ is also negligible, completing the proof. □

Proof of Theorem 3.3. We will apply Lemma 3.4 to subsets of fibres in random lifts of G. Pick vertex z of G, and two walks P_1, P_2 from z to z. Since $|E| > |V|$, we can choose P_1 and P_2 which each contain an edge the other does not contain, and traverse said edge exactly once.

Consider a random lift $\tilde{G}$ of G with fibre size n, constructed by assigning independent random permutations in S_n to the directed edges of G. For $i \in \{1, 2\}$, let σ_i be the product of the permutations along the edges of P_i, where σ_i is the inverse of the permutation assigned to an edge if P_i runs against the direction of the edge. Let $W_i(k)$ be the unique path from (z, k) to $(z, \sigma_i(k)$ lifting P_i. The construction of $\tilde{G}$ guarantees that the $W_i(k)$ are disjoint.

By the assumption on P_1, P_2, the permutations σ_1, σ_2 are distirbuted uniformly and independently. We claim that for $\xi = \frac{\varepsilon(1-\varepsilon)}{2|V(G)|}$, where ε satisfies the conditions of Lemma 3.4 and is at most $\frac{1}{4}$, the graph $\tilde{G}$ is at least a ξ-expander. To this end, let T be set of vertices of $\tilde{G}$ of size at most $\frac{1}{2}|V(\tilde{G})|$. We wish to prove that $|E(T, \bar{T})| \ge \xi|T|$.

For each vertex v of G, let $\tilde{G}_v$ be the set of vertices of $\tilde{G}$ lying above v. Let $T_v = T \cap \tilde{G}_v$, and let $t_v = |T_v|$ and $m = \max_v t_v$. Then we have $|T| \le m|V(G)|$. We consider two cases.

- Suppose for some $u \in V(G)$ we have $t_u < (1-\varepsilon)m$, and take v achieving $t_m = m$. Consider a path Q from v to u in G. There are n disjoint paths in $\tilde{G}$ starting in $\tilde{G}_v$ and ending at $\tilde{G}_u$, at least εm of which contribute to $E(T, \bar{T})$ because at least εm vertices in $\tilde{G}_u$ not in T_u are connected to a vertex in T_v. Since $\frac{\varepsilon}{|V(G)|} \ge \xi$, this finishes.

- If all t_u are at least $(1-\varepsilon)m$, then $|T| \ge (1-\varepsilon)m|V(G)| \ge \frac{3}{4}m|V(G)|$ and $|T| \le \frac{1}{2}n|V(G)|$, so $m \le \frac{2}{3}n$. Let A be the set of $k \in [n]$ with $(z, k) \in T_z$. (Recall that z was the endpoint of the paths P_1 and P_2.) We have $|A| = |T_z| \le m < \frac{2}{3}n$. So as n tends to infinity, almost surely $|A \cup \sigma_1(A) \cup \sigma_2(A) \ge (1+\varepsilon)|A|$. There are at least $\frac{\varepsilon}{2}|A|$ indices $k \in A$ with (without loss of generality) $\sigma_1(k) \notin A$, and for these indices $W_1(k)$ goes from a point in T to a point not in T, and thus contains an edge in $E(T, \bar{T})$. Thus

$$|E(T, \bar{T})| \ge \frac{\varepsilon}{2}|A| \ge \frac{\varepsilon(1-\varepsilon)}{2} \cdot m,$$

whence $\tilde{G}$ is at least a ξ-expander because $|T| \le m|V(G)|$.

Thus random lifts of arbitrary graphs are expanders, as desired. □

We observe that certain special cases of the graph lifting construction give classes of graphs which are of interest in their own right. For example, lifts of a graph consisting of a single vertex and d self-loops are $2d$-regular. In [11], it was shown that the resulting distribution over $2d$-regular graphs is contiguous to the uniform distribution. It follows that random d-regular graphs are expanders, recovering the result of Bollobás [5]. Rather technical analysis give the following result (equivalent to that of Bollobás).

Theorem 3.5 ([2, Theorem 3.1])**.** *Let ε be small enough that $H(\frac{\varepsilon}{2d}) < \frac{d-1}{d} \cdot \log(2)$. Then almost every $2d$-regular graph is an ε-expander.*

Another interesting application, not considered by Amit and Linial, is to lifts of complete graphs. Fix a positive integer $m > 3$, so that K_m satisfies the hypothesis of Theorem 3.3. Then for large n, almost all lifts of K_m of order n should be expanders. We now investigate in more detail the the expansion properties of lifts of K_n.

Conjecture 3.6 (S.A., Y.K., J.V)**.** *For a fixed integer $m > 3$, as n gets large, almost every order-n lift of K_m has edge expansion at least $(1 - o_m(1)) \cdot \frac{m-1}{2}$.*

Progess towards Conjecture. Let $\tilde{K}_m$ denote the order-n lift of K_m. Let $S_1, \ldots, S_m$ denote the sets of vertices of $\tilde{K}_m$ corresponding to each vertex of K_m. Note that $S_1, \ldots, S_m$ partition $V(\tilde{K}_m)$ and $|S_1| = \cdots = |S_m| = n$. Fix a cut (T, U) of $\tilde{K}_m$ where $|T| \le \frac{mn}{2}$. We will show that almost all edge assignments of $\tilde{K}_m$ result in a large conductance across this cut.

For all $i \in [m]$, define $k_i = |S_i \cap T|$. Also, for all $i, j \in [m]$ where $i \ne j$, define the random variable d_{ij} to be the number of edges between $S_i \cap T$ and $S_j \cap U$. The number of edges crossing our cut is precisely $\sum_{i,j} d_{ij}$, so it suffices to lower-bound the d_{ij}.

Let's focus on d_{12}. Note that d_{12} depends only on the set of edges between S_1 and S_2, which ranges uniformly over perfect matchings between S_1 and S_2. This motivates constructing a coupling between A) the uniform distribution over bijective functions f (i.e. perfect matchings) between S_1 and S_2 and B) the uniform distribution over arbitrary functions g from S_1 to S_2. We sample (f, g) as follows.

- Loop through the vertices u of S_1 in any order where the vertices $S_1 \cap T$ appear first.
  - Sample a vertex v of S_2 uniformly at random.
  - Assign $g(u) = v$.
  - Assign $f(u) = v$ *only if doing so would not violate the injectivity of* f. If it would, then resample $f(u)$ until it doesn't.

It's easy to see that this is a valid coupling. Moreover, if $g(u) = v$, then either A) $f(u)$ was also assigned to v or B) $f(u)$ couldn't be assigned to v, meaning $f(u') = v$ for some u' earlier in the ordering than u. In particular,

$$f(S_1 \cap T) \supseteq g(S_1 \cap T).$$

Therefore,

$$d_{12} = |f(S_1 \cap T) \cap (S_2 \cap U)| \geq |g(S_1 \cap T) \cap (S_2 \cap U)|.$$

But notice that the expression on the right is simply the sum of k_1 independent Bernoulli($\frac{n-k_2}{n}$) variables. So by the Chernoff Bound,

$$\Pr\left[d_{12} \leq \frac{k_1(n-k_2)}{n} - \sqrt{2k_1 \ln m}\right] \leq \exp(-4 \ln m) = m^{-4}.$$

The same reasoning applies to all d_{ij}. Therefore, by the union bound, we have with probability $1 - o_m(1)$ that

$$\begin{aligned}
\sum_{\substack{i,j \in [m] \\ i \neq j}} d_{ij} &\geq \sum_{\substack{i,j \in [m] \\ i \neq j}} \left(\frac{k_i(n-k_j)}{n} - \sqrt{2k_i \ln m}\right) \\
&= \left(\sum_{\substack{i,j \in [m] \\ i \neq j}} k_i\right) - \frac{1}{n}\left(\sum_{\substack{i,j \in [m] \\ i \neq j}} k_i k_j\right) - \sqrt{2 \ln m}\left(\sum_{\substack{i,j \in [m] \\ i \neq j}} \sqrt{k_i}\right) \\
&\geq (m-1)\left(\sum_{i \in [m]} k_i\right) - \frac{1}{n}\left(\left(\sum_{i \in [m]} k_i\right)^2 - \sum_{i \in [m]} k_i^2\right) - \sqrt{2m^2 \ln m}\left(\sum_{i \in [m]} \sqrt{k_i}\right) \\
&\geq (m-1)|T| - \frac{1}{n}\left(|T|^2 - \frac{|T|^2}{m}\right) - \sqrt{2m^3 \ln m}\sqrt{|T|}.
\end{aligned}$$

The conductance is thus at least

$$\begin{aligned}
\frac{1}{|T|} \sum_{\substack{i,j \in [m] \\ i \neq j}} d_{ij} &\geq m - 1 - \left(\frac{1}{n} - \frac{1}{mn}\right)|T| - \frac{\sqrt{2m^3 \ln m}}{\sqrt{|T|}} \\
&\geq m - 1 - \left(\frac{1}{n} - \frac{1}{mn}\right)\left(\frac{mn}{2}\right) - \frac{\sqrt{2m^3 \ln m}}{\sqrt{|T|}} \\
&= \frac{m-1}{2} - \frac{\sqrt{2m^3 \ln m}}{\sqrt{|T|}}.
\end{aligned}$$

When $|T| = \omega_m(m \ln m)$, we recover the claimed bound. We do not know how to proceed in the case where $|T|$ is small, but we believe a similar agument may work. $\square$

We believe the methods of both Conjecture 3.6 can be extended to any base graph, albeit with more technicalities. This gives a more elementary understanding of Theorem 3.3, the original proof of which involved technical analysis of asymptotics of the entropy function.

4 Geometric Construction

In this section, we focus on a stronger notion of expansion.

Definition 4.1. We say a graph $G = (V, E)$ is a *two-dimensional spectral expander* if

- G itself is a spectral expander,
- for all $v \in V$, the induced subgraph of G on $N(v)$ is a spectral λ-expander for some $\lambda < \frac{1}{2}$.

Remark 4.2. The second condition implies the first by the so-called Trickling Down Theorem [16].

Most constructions of sparse expanders described earlier do not exhibit local expansion. For example, in random d-regular graphs (which are expanders by Theorem 3.5) and sparse Erdos-Renyi graphs (which are expanders by Problem Set 2), the neighborhood of any vertex almost surely contains no cycles (and hence cannot be an expander). Some algebraic constructions (in the vein of [1]) exhibit local expansion, but these are often tied to specific properties of the groups that generate them. It is therefore interesting to ask whether simpler constructions exist.

Liu et al. [13] answer in the affirmative by providing a natural geometric family of two-dimensional spectral expanders.

Definition 4.3. We define the "geometric" distribution $\mathrm{Geo}_d^2(n, p)$ on graphs G on $[n]$ by the following sampling procedure.

- Independently sample $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \sim \mathrm{Unif}\{\mathbb{S}^{d-1}\}$.
- For all $i, j \in [n]$ where $i \neq j$, add an edge between i and j if and only if

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \geq \tau,$$

  where τ is a constant chosen so that the marginal probability of each edge existing is p.

Theorem 4.4 (Main Result)**.** *For all $0 < \epsilon < 1$, there exist constants C, δ such that for $H \sim \mathrm{Geo}^2_{C \log n}(n, n^{-1+\epsilon})$, all vertex links of H are spectral $(\frac{1}{2} - \delta)$-expanders with high probability.*

At a high level, Liu et al. [13] prove this theorem using two steps.

- First, they show that if the Markov Chain over a graph mixes rapidly, then a random restriction of the graph is a spectral expander with high probability. This is essentially a converse to the type of result shown in [8].
- Next, they show that the Markov Chain over an *infinite* geometric graph mixes rapidly.

Each vertex link of a random geometric graph on $\mathbb{S}^{d-1}$ is approximately[1] equivalent to a random geometric graph on $\mathbb{S}^{d-2}$ (because most points in a spherical cap will be distributed near the boundary, which is the surface of a lower-dimensional sphere). Moreover, every random geometric graph on $\mathbb{S}^{d-2}$ can be viewed as a random restriction of the infinite geometric graph on $\mathbb{S}^{d-2}$. Therefore, these two results together essentially imply the main result.

We state these more precisely below.

Theorem 4.5. *Let X be a (possibly infinite) vertex-transitive graph with stationary distribution ρ. Sample n vertices independently from X according to ρ and let G be the subgraph induced by them. Let p be the marginal probability that each edge appears in G. Let A_G and $\hat{A}_G$ denote the adjacency matrix and the normalized adjacency matrix of G respectively. Suppose there exist constants $C \geq 1$ and $\lambda \in [\frac{1}{\sqrt{np}}, 1]$ such that*

- $\|\alpha X^k - \rho\|_{TV} \leq C\lambda^k$ *for all distributions α and positive integers k,*
- $pn >> C^6 \log^4 n$.

Then

$$\Pr\left[|\lambda_2(\hat{A}_G)| \leq (1 + o(1)) \cdot \max\left(\lambda, \tfrac{\log^4 n}{\sqrt{pn}}\right)\right] \geq 1 - n^{-\lambda} \quad \forall \gamma > 0.$$

[1] Liu et al.[13] do significant work to make this rigorous; we omit the details here for simplicity.

Theorem 4.6. *Consider the following Markov Chain on $\mathbb{S}^{d-1}$. From a point $\boldsymbol{v} \in \mathbb{S}^{d-1}$, we move to a uniformly random point $\boldsymbol{u}$ in the spherical cap defined by $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq \tau$, where τ is chosen so that the spherical cap has measure p.*

Let P_p be the transition operator of this Markov Chain and let ρ be its stationary distribution (which is uniform). Then, for all distributions α over $\mathbb{S}^{d-1}$, we have

$$\|\alpha P_p^k - \rho\|_{TV} \leq ((1 + o_{d\tau^2}(1)) \cdot \tau)^{k-1} \cdot \sqrt{\frac{1}{2} \log \frac{1}{p}}.$$

In the remainder of this section, we describe the proofs of these two theorems. Theorem 4.5 is purely linear algebraic, and relies on the following fact.

Lemma 4.7. *For any $n \times n$ symmetric matrix M and rank-1 positive semidefinite matrix R, we have $|\lambda_2(M)| \leq \|M - R\|$, where $\| \, \|$ is the operator norm.*

We now give a sketch of the proof of Theorem 4.5 which will illuminate the connection between Markov Chain mixing and spectral expansion.

Proof of Theorem 4.5.
Let D_G be the diagonal matrix whose entries are the degrees of the vertices of G. Note that the normalized adjacency matrix $\hat{A}_G$ can be expressed as $D_G^{-1/2} A_G D^{-1/2}$. Define $R = pD_G^{-1/2} J D_G^{-1/2}$, where J is the all-ones matrix. Check that R is rank-1 and positive semidefinite. Now, using Lemma 4.7 and the fact that the operator norm is sub-multiplicative and sub-additive, we get

$$\begin{aligned}
|\lambda_2(\hat{A}_G)| &\leq \|\hat{A}_G - R\| \\
&= \|D_G^{-1/2} A_G D_G^{-1/2} - pD_G^{-1/2} J D_G^{-1/2}\| \\
&= \|D_G^{-1/2} (A_G - pJ) D_G^{-1/2}\| \\
&\leq \|D_G^{-1/2}\| \cdot \|(A_G - pJ)\| \cdot \|D_G^{-1/2}\| \\
&= \|D_G^{-1/2}\|^2 \cdot \|(A_G + pI - pJ) - pI\| \\
&\leq \|D_G^{-1/2}\|^2 \cdot (\|(A_G + pI - pJ)\| + p).
\end{aligned}$$

We bound these terms separately:

- The operator norm of $D_G^{-1/2}$ is simply $d_{min}^{-1/2}$, where d_{min} is the minimum degree of G. Note that each degree of G is the sum of n independent Bernoulli variables with parameter p. Therefore, by Chernoff-Hoeffding, the probability that any vertex has degree less than $pn - \sqrt{pn \log^2 n}$ is at most

$$\exp\left(- \frac{2\left(\sqrt{pn \log^2 n}\right)^2}{n} \right) = n^{-p \log n}.$$

  Hence, by the union bound, all n vertices have degrees greatee than $pn - \sqrt{pn \log^2 n}$ with probability $n^{-\omega(1)}$. In particular, this means $d_{min} > pn - \sqrt{pn \log^2 n}$, so

$$\|D_G^{-1/2}\|^2 = d_{min}^{-1} \leq \frac{1}{pn - \sqrt{pn \log^2 n}} \approx \frac{1}{pn}\left(1 + \frac{\log n}{\sqrt{pn}}\right).$$

- In order to bound the operator norm of $A_G + pI - pJ$, we use the technique introduced in Section 2 of instead bounding

$$\mathbb{E}\left[\mathrm{tr}\left((A_G + pI - pJ)^\ell\right)\right].$$

  Note that $A_G + pI - pJ$ is the adjacency matrix of a complete graph where edges have weight $1 - p$ if they are in G and $-p$ if they are not in G. The term $\mathrm{tr}((A_G + pI - pJ)^\ell)$ therefore counts closed walks of length ℓ on this graph, weighted by the product of their edge weights.

  This explains the relevance of the mixing speed of random walks on X. It turns out that the quantity $\|\alpha X^t - \rho\|_{TV}$, for a suitable distribution α, is closely tied to the aforementioned sum over closed walks. The precise details are extremely complicated and computation-heavy, so we omit them here.

□

We now turn to Theorem 4.6. We first introduce some notation to more easily discuss distributions on the sphere:

Definition 4.8. A distribution α on $\mathbb{S}^{d-1}$ is *symmetric about* $y \in \mathbb{S}^{d-1}$ if there exists a function $\ell_a : [-1, 1] \to \mathbb{R}$ such that $\alpha(z) = \ell_a(\langle z, y\rangle)$, We say α is *spherically monotone* if the function ℓ_a is nondecreasing.

Definition 4.9. We define ρ to be the uniform distribution on $\mathbb{S}^{d-1}$. The corresponding function ℓ_ρ is constant.

Definition 4.10. Given a distribution α on $\mathbb{S}^{d-1}$, we let P_α be the transition operator of the associated Markov Chain.

The key idea in the proof of Theorem 4.6 is to relate the given Markov Chain to Brownian Motion on $\mathbb{S}^{d-1}$, which is already known to converge rapidly to the uniform distribution ([3], [9]).

Specifically, we will use the following lemma to relate the Markov Chain with Brownian Motion:

Lemma 4.11. *Let μ, ν be spherically monotone distributions on $\mathbb{S}^{d-1}$ centered at y such that $\ell_\nu \preceq_{st} \ell_\mu$ Then*

1. $\|\ell_\nu - \ell_\rho\|_{TV} \leq \|\ell_\mu - \ell_\rho\|_{TV}$,
2. $\ell_{P_\nu \alpha} \preceq_{st} \ell_{P_\mu \alpha}$ *for all spherically monotone* α.

Intuitition Behind Lemma .

Since ℓ_μ stochastically dominates ℓ_ν, the mass of ℓ_μ is larger than that of ℓ_ν in any cap. So it's clear that ℓ_μ should be further away from the uniform distribution, showing (1).

For (2), note that $P_\nu\alpha = P_\alpha\nu$ and $P_\mu\alpha = P_\alpha\mu$ because convolving distributions is commutative. Now consider the coupling between $v \sim \nu$ and $u \sim \mu$ where u is guaranteed to be closer to y than v (this must exist by the definition of stochastic dominance). Intuitively, the resulting value of $P_\alpha u$ should be concentrated closer to y than $P_\alpha v$. This shows that $\ell_{P_\alpha \nu} \preceq_{st} \ell_{P_\alpha \mu}$. □

By the lemma, it suffices to establish that Brownian Motion (over a certain time interval) stochastically dominates the uniform distribution over a cap of measure p. Again, the full proof is complicated and has many parts. So we will instead try to capture the key details by proving a weaker result: that Brownian Motion *concentrates* in a cap.

Lemma 4.12. *Let $(V_t)_{t\geq 0}$ be Brownian Motion on $\mathbb{S}^{d-1}$ starting at V_0. Then for any time $t \geq 0$,*

$$\Pr[\langle V_0, V_t\rangle - \exp(-(d-1)t)| \geq x] \leq 2\exp\left(-\frac{d-1}{2} \cdot \frac{x^2}{1 - e^{-2(d-1)t}}\right).$$

Proof. Let $(B_t)_{t\geq 0}$ be standard Brownian Motion. Then Brownian Motion on $\mathbb{S}^{d-1}$ satisfies

$$dV_t = \sqrt{2}(\mathbb{1} - V_t V_t^T)dB_t - (d-1)V_t dt.$$

Define $A_t = \langle V_0, V_t\rangle$. After some omitted algebra, we get

$$dA_t = -\theta \cdot A_t dt + \sqrt{2}\sqrt{1 - A_t^2} dB_t.$$

This motivates the substitution $A_t = R_t + \exp(-\theta t)$. After some more manipulations, this gives

$$dR_t = -\theta R_t dt + \sqrt{2}\sqrt{1 - A_t^2} dB_t.$$

We can simplify this even further by computing

$$d(R_t \exp(\theta t)) = \sqrt{2}\exp(\theta t)\sqrt{1 - A_t^2} dB_t.$$

This differential equation says that R_t is a process whose (infinitesimal) transitions are bounded. Thus, we may apply a continuous version of Azuma-Hoeffding to bound $|R_t|$; the details are cumbersome, but we obtain the claimed expression. □

We tie everything together below:

Sketch of Theorem 4.6. Lemma 4.12 can be used to show that the transition operator of Brownian Motion on $\mathbb{S}^{d-1}$ in a certain time interval stochastically dominates by the Markov Chain P_p. Then Lemma 4 shows that the mixing speed of P_p is at most that of Brownian Motion, which has a known bound. □

5 Experimental Findings

In this section, we give some experimental results to give more intuition for the results outlined in this paper. The code for our simulations can be found here:

`https://drive.google.com/drive/folders/1nLh7DwGgBygqXmlCAIPx3j5Ar28SUUoZ`

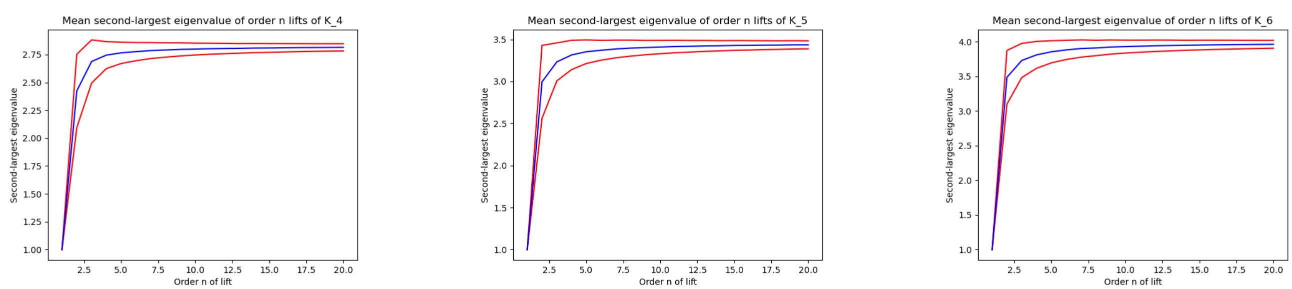
5.1 Lifts of Complete Graphs

Conjecture 3.6, which we study with elementary edge-expansion methods, can also be understood through the lens of spectral graph theory, used elsewhere in this report to analyze various expander constructions. Namely, a graph with high edge expansion has normalized adjacency matrix with low second-highest eigenvalue. The adjacency graph of a lift of K_m is easy to describe: it is a symmetric $mn \times mn$ matrix of $n \times n$ blocks, where each block except the m on the diagonal is a random permutation matrix. Each n-lift of K_m is $(m-1)$-regular, so the normalized adjacency matrix is given by dividing the adjacency matrix by $(m-1)$.

To provide further intuition for Conjecture 3.6, we describe some empirical findings on spectral expansion of random n-lifts on K_m. First, we discuss the spectral expansion of n-lifts of K_m for a fixed m, as n grows large. Figure 1 shows that the expected spectral expansion of random n-lifts K_m approaches a constant as n approaches infinity, and that the standard deviation of the distribution of second-largest eigenvalues decreases. Thus, with increasingly high probability as n grows large, the spectral expansion of a random n-lift of K_m should be close to some constant less than 1 (this follows from normalizing the limit), providing empirical support for the statement of Conjecture 3.6 that n-lifts of K_m are expanders.

Conjecture 3.6 states further that random lifts of K_m get better as m gets large. Translating into the language of spectral expansion, this means that the second-largest eigenvalue of the normalized adjacency matrix of a random n-lift of K_m, with n large, should decrease as m gets large. This is demonstrated in Figure 2.

Finally, it is interesting to investigate the distribution of second-largest eigenvalues independently of applications to edge expansion. Figure 3 shows this distribution for K_4 and several values of n. It is interesting to note that the distribution is not symmetric; in particular, for small n, there is a noticeable spike at $\lambda_2 = 3$, which corresponds to the case when the random lift is not connected. This spike decreases in size rapidly as n grows, reflecting the fact that almost all n-lifts of a graph are connected. The graphs also confirm that the distribution does because more concentrated as n increases, further corroborating Conjecture 3.6. Figure 4 shows the distribution of second-largest eigenvalues of random 20-lifts of K_n, again revealing an asymmetric distribution. It would be interesting to study the distribution of random n-lifts of K_m for general m and n with aim of better understanding the concentration and asymmetry of the distribution.



(a) $m = 4$ (b) $m = 5$ (c) $m = 6$

Figure 1: Mean (unnormalized) second-largest eigenvalue of random n-lift of K_m over 10000 samples, with $n \leq 20$. Blue line shows mean; red line shows error bars of one standard deviation on each side.

5.2 Geometric Construction

We experimentally investigate the properties of the geometric construction of expander graphs.

Initially, we apply Markov Chain Monte Carlo to the Markov Chain outlined in 4.6 in order to observe its rapid mixing behavior. We present the outcomes when choosing $n = 1000$ and $n = 10000$ points, respectively, with the Markov Chain being simulated for $t = 100$, in Figure 5 and Figure 6. The graph visualizes how the distribution rapidly mixes to the stationary distribution, which is uniform on the sphere.

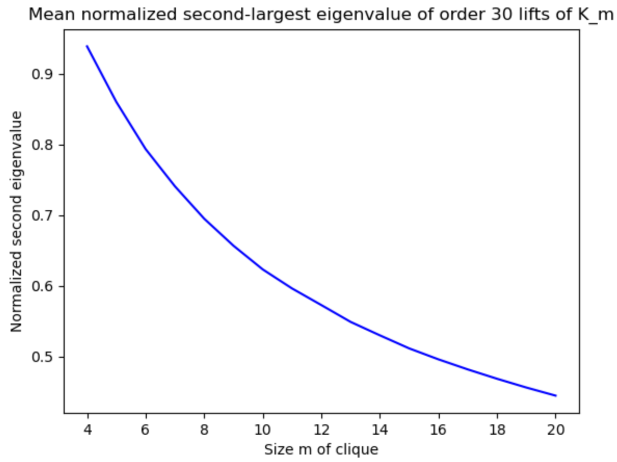
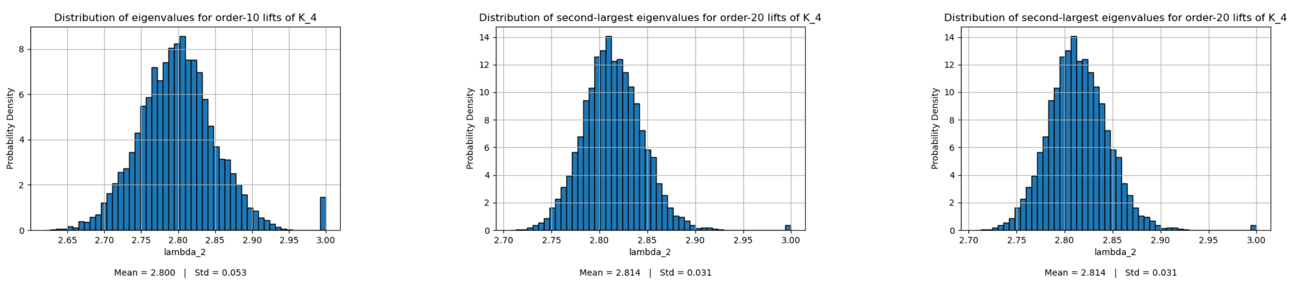


Figure 2: Mean normalized second-largest eigenvalue of random 30-lift of K_m over 100 samples.



(a) $n = 10$ (b) $n = 20$ (c) $n = 30$

Figure 3: Distribution of (unnormalized) second-largest eigenvalue of random n-lift of K_4 over 10000 samples, with $n \in \{10, 20, 30\}$.

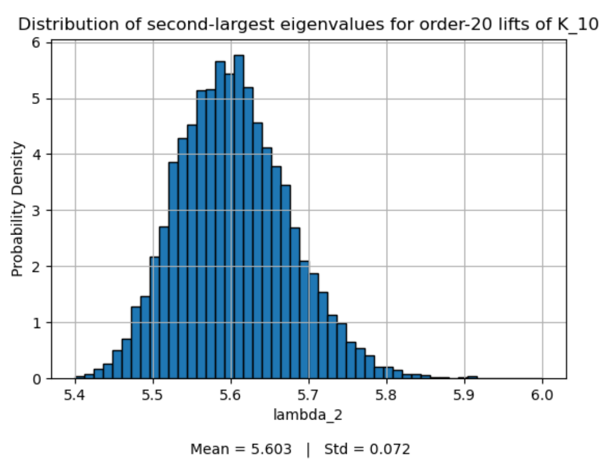


Figure 4: Distribution of (unnormalized) second-largest eigenvalue of random 20-lift of K_{10} over 10000 samples.

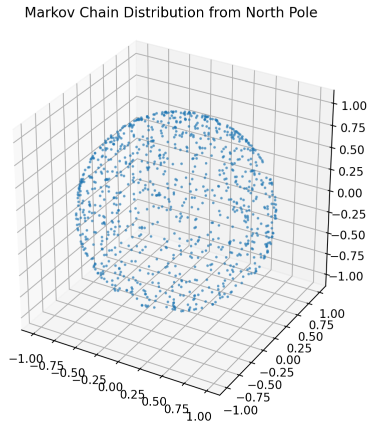


Figure 5: $n = 1000$

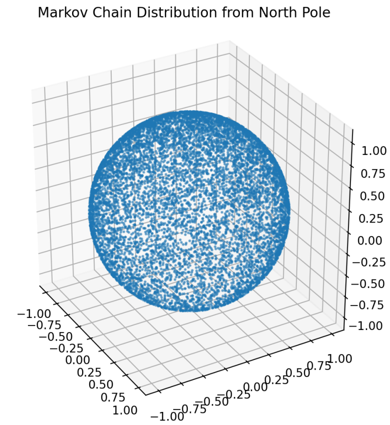


Figure 6: $n = 10000$

In examining Theorem 4.4, we experimented with the case where $C = 20$ and $\epsilon = 0.4$ for $n = 2000, 2100, \ldots, 2900$. A spectral gap exceeding $1/2$ was noted for every n, as seen in Figure 7.

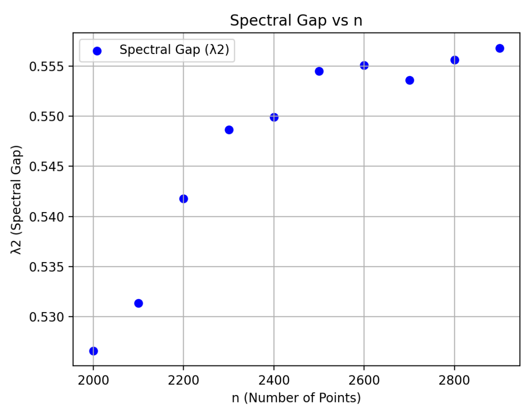


Figure 7: Spectral gap for random graphs sampled in $\mathrm{Geo}_{C \log n}(n, n^{1+\epsilon})$ with $n = 2000, 2100, \ldots, 2900$.

We did not test the main theorem for arbitrarily large n, but the example cases do show that the theorem holds for C, ϵ for some n.

6 Conclusion

In this paper, we survey several constructions of families of expander graphs, using algebraic, combinatorial, and geometric methods. We posit Conjecture 3.6, which gives a more intuitive understanding of the combinatorial underpinnings of Theorem 3.3; thus far, our analysis of this conjecture has not required the technical analyses of the entropy function that the proof of Theorem 3.3 required. We also experimentally study the lifting construction for graphs and the geometric construction of expanders, giving empirical support for our conjecture and providing further intuition behind these constructions.

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